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Stability of a Two-Dimensional Jet

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Calculations are presented for the stability of a two-dimensional jet. Since the jet spreads, initially very rapidly, a parallel-flow approximation may not be used. The method of multiple scales is used to account for the flow divergence. The growth of axial velocity fluctuations in the jet is examined as a function of axial and transverse location. Neutral curves are presented on the basis of various definitions of the neutral points. The results are compared with parallel flow calculations and inclusion of the flow divergence is found to predict increased growth rates. Various features of the calculations indicate that the multiple scales technique has only a limited range of validity, and an analysis is presented that determines this range.

I. Introduction

IN spite of the fact that many flows of practical interest, such as boundary layers, jets, and wakes, are nonparallel, linear stability analyses historically have used a quasiparallel mean flow approximation. Although this approximation might at first glance appear reasonable for bounded flows where the critical Reynolds number is high, it is clearly less justifiable for free shear flows where experiments show the flow to be unstable at very low Reynolds numbers where the basic flow is strongly divergent. Various intuitive approaches to account for the effects of flow divergence, e.g., Cheng[1] and Ko and Lessen,[2] have been adopted. Formal expansions of the linear stability problem about some axial location have been obtained by Lanchon and Eckhaus[3] and by Ling and Reynolds.[4] In the latter case numerical calculations were performed for the problems of the Blasius boundary layer, the two-dimensional jet, and the two-dimensional flat-plate wake. However, their expansion in powers of ϵ, a small parameter depending on the flow under study, e.g., $\epsilon=(x_0U_\infty/\nu)^{-1/2}$ for the Blasius boundary layer, where x_0 was some axial location, breaks down when $(x-x_0)=0(\epsilon^{-1})$.

Alternative expansion schemes have been introduced by Bouthier,[5,6] Gaster,[7] Saric and Nayfeh,[8] and Eagles and Weissman[9] which eliminate these difficulties and describe the cumulative effects of divergence of the flow. The method used in the present work makes use of the method of multiple scales that is described in detail by Nayfeh.[10]

In the present work the linear stability of a two-dimensional jet flow is examined. A similar analysis has been performed by Garg and Round,[11] however, as will be described later in this paper, there are several omissions in their work. It is recognized that since the rate of spread of the jet in the region of instability is relatively high it is questionable whether the use of an expansion scheme that uses the spread rate as the expansion parameter is justified without the inclusion of many terms in the expansion. Thus, it is to be expected that the numerical results obtained in this paper will only provide a qualitative assessment of the effects of flow divergence. However, it should be noted that in another use of the method of multiple scales where the expansion parameter was not very small, Kaiser and Nayfeh[12] have shown good agreement between the multiple scales solution and wave-envelope and weighted-residual techniques for sound propagation in nonuniform ducts up to a wall slope of 0.2 with good qualitative agreement up to a slope of 0.4.

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In the present paper numerical results are presented for the axial growth and decay of fixed real frequency instability waves propagating in the two-dimensional jet flow. The relationship between these results and those of parallel flow stability theory is examined. This leads to a discussion of the range of validity of the multiple-scales technique as it is presently applied in linear stability theory.

II. Formulation and Method of Solution

Since the basic analysis used in this paper has been developed in detail by Saric and Nayfeh,[8,13] only a brief outline, necessary for subsequent development, will be given. The boundary-layer equations for the primary flow are nondimensionalized with respect to velocity and length scales u_r^* and δ_r^*, respectively, where

$$u_r^*=2\alpha^{*2}/3x_0^{*1/3} \tag{1a}$$

and

$$\delta_r^*=3\nu^{*1/2}x_0^{*2/3}/\alpha^* \tag{1b}$$

where the asterisk denotes dimensional quantities, x_0^* is an arbitrary axial location, and ν^* is the kinematic viscosity. α^* is a parameter related to the axial momentum flux of the jet. From Schlichting[14]

$$J^*=16\rho^*\alpha^{*3}\nu^{*1/2}/9 \tag{2a}$$

where

$$J^*=\rho^*\int_{-\infty}^{+\infty}u^{*2}\mathrm{d}y^* \tag{2b}$$

and u^* is the axial velocity. The analysis will assume, subject to the reservations just discussed, that the primary flow is slowly diverging with axial distance x. The stream function for the primary flow may then be written,

$$\Psi=\Psi(x_1,y), \qquad x_1=\epsilon x \tag{3}$$

where x_1 is a slow axial scale and ϵ a parameter characterizing the divergence of the primary flow. In the present problem ϵ is given by

$$\epsilon=1/R=\nu^*/u_r^*\delta_r^* \tag{4}$$

where R is a Reynolds number. The stream function of the primary flow is presumed known from a solution of the Navier-Stokes equations. If the effects of displacement thickness are taken to be of $0(\epsilon^2)$ the stream function for the primary flow, including terms of order ϵ, may be written[14]

$$\Psi(x_1,y)=(6x_1)^{1/3}\tanh\xi, \qquad \xi=y/(6x_1)^{2/3} \tag{5}$$

so that the velocity components of the primary flow may be written,

$$\frac{\partial\Psi}{\partial y}=U=U_0(x_1,y)+\epsilon U_1(x_1,y)+\cdots \tag{6a}$$

and

$$-\frac{\partial \Psi}{\partial x} = -\epsilon \frac{\partial \Psi}{\partial x_1} = \epsilon V(x_1, y) + \cdots \tag{6b}$$

where for the two-dimensional jet

$$U_0 = (6x_1)^{-1/3} \operatorname{sech}^2 \xi, \qquad U_1 = 0 \tag{7a}$$

and

$$V = 2(6x_1)^{-2/3}(2\xi \operatorname{sech}^2 \xi - \tanh \xi) \tag{7b}$$

The perturbation stream function $\psi(x,y,t)$ is expanded in a series of the form

$$\psi(x,y,t) = \left[\sum_{n=0}^{\infty} \epsilon^n \phi_n(x_1,y) \right] \exp[i\theta(x,t)] \tag{8}$$

where

$$\frac{\partial \theta}{\partial x} = k_0(x_1) \text{ and } \frac{\partial \theta}{\partial t} = -\omega \tag{9}$$

In the present analysis we will consider the spatial stability of the jet so that ω is a fixed real frequency. Substituting expansions of the form Eqs. (6) and (8) into the Navier-Stokes equations for the perturbation stream function and equating coefficients of like powers of ϵ gives the following.

Order ϵ^0:

$$\mathcal{L}(\phi_0) = \left(\frac{\partial^2}{\partial y^2} - k_0^2 \right)^2 \phi_0 - ik_0 R \left[\left(U_0 - \frac{\omega}{k_0} \right) \times \left(\frac{\partial^2 \phi_0}{\partial y^2} - k_0^2 \phi_0 \right) - \frac{\partial^2 U_0}{\partial y^2} \phi_0 \right] = 0 \tag{10}$$

$$\phi_0 \to 0 \text{ as } y \to \pm \infty \tag{11}$$

Order ϵ:

$$\mathcal{L}(\phi_1) = \chi_1(\phi_0, U_0, V, k_0) \tag{12}$$

$$\phi_1 \to 0 \text{ as } y \to \pm \infty \tag{13}$$

The function χ_1 is given by Saric and Nayfeh.[8] Since x_1 appears only parametrically in Eq. (10) ϕ_0 may be written as

$$\phi_0(x_1, y) = A_0(x_1) \zeta(x_1, y) \tag{14}$$

where $A_0(x_1)$ is an undetermined amplitude function and ζ satisfies the Orr-Sommerfeld problem

$$\mathcal{L}(\zeta) = 0 \tag{15}$$

$$\zeta \to 0 \text{ as } y \to \pm \infty \tag{16}$$

Substituting Eq. (14) into Eq. (12) gives

$$\mathcal{L}(\phi_1) = R\mathfrak{M}(A_0, \zeta, U_0, V, k_0) \tag{17}$$

The unknown function $A_0(x_1)$ may then be determined by applying the solvability condition to Eq. (17) which states that the inhomogeneous Eq. (17) only has a solution if

$$\int_{-\infty}^{\infty} \mathfrak{M} \zeta^* \mathrm{d}y = 0 \tag{18}$$

where ζ^* is the solution of the adjoint homogeneous problem

$$\mathcal{L}^*(\zeta^*) = \left(\frac{\partial^2}{\partial y^2} - k_0^2 \right)^2 \zeta^* - ik_0 R \left[\left(U_0 - \frac{\omega}{k_0} \right) \times \left(\frac{\partial^2 \zeta^*}{\partial y^2} - k_0^2 \zeta^* \right) + 2 \frac{\partial U_0}{\partial y} \frac{\partial \zeta^*}{\partial y} \right] = 0 \tag{19}$$

$$\zeta^* \to 0 \text{ as } y \to \pm \infty \tag{20}$$

Substituting for $\mathfrak{M}$ from Eq. (17) into Eq. (18) provides an equation for the axial development of $A_0(x_1)$ in the form

$$\frac{\mathrm{d}A_0}{\mathrm{d}x_1} = ik_1 A_0 \tag{21}$$

where the form of k_1 is readily obtained and is given by Saric and Nayfeh[8] in their Eq. (26). This completely defines the zero-order solution for the perturbation stream function, $\phi_0(x_1, y)$.

The two-dimensional jet is known to be unstable to both symmetric and antisymmetric disturbances.[15] Since the former are the least stable, calculations will only be provided for this mode, so that the boundary conditions for ζ, Eq. (16), may be replaced by

$$\frac{\partial \zeta}{\partial y} = \frac{\partial^3 \zeta}{\partial y^3} = 0 \text{ at } y = 0 \tag{22a}$$

and

$$\zeta \to 0 \text{ as } y \to \infty \tag{22b}$$

with corresponding boundary conditions for the adjoint eigenfunction ζ^*.

In order to evaluate k_1 in Eq. (21) it is necessary to calculate $\partial \zeta / \partial x_1$ and $\mathrm{d}k_0/\mathrm{d}x_1$. These values may be obtained at a given axial location by first differentiating Eq. (10) with respect to x_1, giving

$$\mathcal{L}\left(\frac{\partial \zeta}{\partial x_1} \right) = h_1 + h_2 \frac{\mathrm{d}k_0}{\mathrm{d}x_1} \tag{23}$$

with

$$\frac{\partial^2 \zeta}{\partial y \partial x_1} = \frac{\partial^4 \zeta}{\partial y^3 \partial x_1} = 0 \text{ at } y = 0 \tag{24a}$$

and

$$\frac{\partial \zeta}{\partial x_1} \to 0 \text{ as } y \to \infty \tag{24b}$$

The functions h_1 and h_2 are given by Saric and Nayfeh.[8] Applying the solvability condition to Eq. (23) enables $\mathrm{d}k_0/\mathrm{d}x_1$ to be calculated from

$$\frac{\mathrm{d}k_0}{\mathrm{d}x_1} = -\left(\int_0^\infty h_1 \zeta^* \mathrm{d}y \right) \Big/ \left(\int_0^\infty h_2 \zeta^* \mathrm{d}y \right) \tag{25}$$

The integration in Eq. (18) and Eq. (25) need only be performed from $y = 0$ to infinity since the functions $\mathfrak{M}$, h_1, and h_2 are all even functions of y. Having obtained the value of $\mathrm{d}k_0/\mathrm{d}x_1$, the inhomogeneous problem, Eqs. (23) and (24), may be solved.

In order to obtain ζ and ζ^* we note that sufficiently far from the jet centerline, say, $y \geq y_m$, we may write

$$U_0 = \frac{\partial U_0}{\partial y} = \frac{\partial^2 U_0}{\partial y^2} = 0; \qquad y \geq y_m \tag{26}$$

The linearly independent solutions to the simplified forms of Eqs. (15) and (19) are

$$\zeta_1 = \zeta_1^* = \exp(-k_0 y) \tag{27a}$$

and

$$\zeta_2 = \zeta_2^* = \exp(-\overline{k_0} y), \qquad \overline{k_0} = [k_0^2 - i\omega R]^{1/2} \tag{27b}$$

Using these asymptotic solutions for the starting conditions at $y = y_m$ Eq. (15) may be integrated for a given value of ω and a guessed value for the eigenvalue k_0. A variable step-size fourth-order Runge-Kutta routine was used to integrate the equations and, since the Reynolds number was relatively small in all the calculations, no orthogonalization routine was necessary to insure the linear independence of the solutions. Both boundary conditions, Eq. (22a), will be satisfied only if k_0 is an eigenvalue. In that case, the eigenfunction is given by

$$\zeta=\zeta_1+c(x_1)\zeta_2 \tag{28}$$

where

$$c(x_1)=-\left[\frac{\partial\zeta_1}{\partial y}(0)\right]\Big/\left[\frac{\partial\zeta_2}{\partial y}(0)\right]=-\left[\frac{\partial^3\zeta_1}{\partial y^3}(0)\right]\Big/\left[\frac{\partial^3\zeta_2}{\partial y^3}(0)\right] \tag{29}$$

The eigenvalue may be obtained by successive integrations where the next estimated value of k_0 is obtained from an inverse Lagrangian interpolation procedure.[16] The adjoint eigenfunction ζ^* and the axial derivative of the eigenfunction, $\partial\zeta/\partial x_1$, may then be readily calculated using the method described by Saric and Nayfeh.[8] Note that in the evaluation of $\partial\zeta/\partial x_1$ we have as starting conditions

$$\left(\frac{\partial\zeta}{\partial x_1}\right)=\left(\frac{\partial\zeta}{\partial x_1}\right)_1+c(x_1)\left(\frac{\partial\zeta}{\partial x_1}\right)_2+\frac{\mathrm{d}c}{\mathrm{d}x_1}\zeta_2,\quad y=y_m \tag{30}$$

where

$$\left[\left(\frac{\partial\zeta}{\partial x_1}\right)_1\right]_{y=y_m}=-\frac{\mathrm{d}k_0}{\mathrm{d}x_1}y_m\exp(-k_0y_m) \tag{31a}$$

$$\left[\left(\frac{\partial\zeta}{\partial x_1}\right)_2\right]_{y=y_m}=-\frac{\mathrm{d}\overline{k_0}}{\mathrm{d}x_1}y_m\exp(-\overline{k_0}y_m) \tag{31b}$$

Equation (23) may then be integrated twice using these starting conditions for $(\partial\zeta/\partial x_1)_1$ and $(\partial\zeta/\partial x_1)_2$ and the complete solution is given by

$$\frac{\partial\zeta}{\partial x_1}=\left(\frac{\partial\zeta}{\partial x_1}\right)_1+c(x_1)\left(\frac{\partial\zeta}{\partial x_1}\right)_2+\frac{\mathrm{d}c}{\mathrm{d}x_1}\zeta_2 \tag{32}$$

The value of $\mathrm{d}c/\mathrm{d}x_1$ is determined by satisfying either of the boundary conditions (24a), noting that the other boundary condition will automatically be satisfied.

The preceding analysis enables the first-order effects of flow divergence on the jet stability to be calculated. Calculations based on this analysis, which will be described, indicated that the solution might not be valid for large values of y. An analysis of the behavior of the solution in this region is provided here in anticipation of the stability calculations.

It is important to note one essential difference between parallel and nonparallel flow calculations. In the former case the amplification rate and wavenumber of a disturbance are independent of the physical variable examined, for example, axial velocity or pressure disturbances, and are independent of transverse distance. Neither of these statements may be made for nonparallel stability calculations. Thus, for example, the growth rate and wavenumber associated with the perturbation stream function to a first approximation, from Eqs. (8) and (14) are seen to be

$$\sigma=\mathrm{Re}\left[\frac{\partial}{\partial x_1}(\ell n\psi)\right]=\mathrm{Re}\left\{ik_0+\epsilon\left[ik_1+\frac{\partial}{\partial x_1}(\ell n\zeta)\right]\right\} \tag{33a}$$

and

$$\alpha=\mathrm{Im}\left[\frac{\partial}{\partial x_1}(\ell n\psi)\right] \tag{33b}$$

respectively. The first term on the right-hand side of Eq. (33a) is the parallel-flow approximation and the remaining terms are associated with the slow axial variation of the first-order perturbation streamfunction, $\phi_0(x_1,y)$. Since $\zeta(x_1,y)$ is a function of y, then the growth rate and wavenumber will also be functions of transverse distance. Similarly, since the transverse variation of each variable considered will be different, so will its growth rate and wavenumber. Clearly, the calculated "stability" of the flow depends on which variable's growth or decay is examined. At this point it is useful to note that the calculations of Saric and Nayfeh[8] and Garg and Round[11] both disregard the contribution of the last term in Eq. (33a) to the growth rate. Saric and Nayfeh[13] argue that for the flat plate boundary layer the numerical results for the growth rate based on ignoring this last term are close to that obtained by considering the complete equation for the growth rate of the axial velocity perturbation at or near its maximum amplitude. However, the relative contributions to the growth rate from the last two terms in Eq. (33a) depends on how the eigensolution $\zeta(x_1,y)$ is normalized. Garg and Round[11] used condition (28) of this paper. Thus, their numerical values for the growth rate may not be associated, without further analysis, with the growth of any physical disturbance in the flow.

Consider the behavior of the asymptotic expansion at large transverse distances. For convenience in the analysis it will be assumed that the disturbance behavior at large values of y, say, $y\geq y_m$, is essentially inviscid, so that

$$\zeta\simeq\exp(-k_0y),\qquad y\geq y_m \tag{34}$$

From Eq. (33a) it can be seen that

$$\sigma=\mathrm{Re}\left[ik_0+\epsilon\left(ik_1-\frac{\mathrm{d}k_0}{\mathrm{d}x_1}y\right)\right],\qquad y\geq y_m \tag{35}$$

Thus, for large values of y the growth rate, and also the wavenumber α, will depend on the value of $\mathrm{d}k_0/\mathrm{d}x_1$. For the present problem this will be seen to result in an increasing growth rate with transverse distance which appears to be physically unrealistic.

In order to examine the validity of the multiple-scales expansion we will consider the behavior of the higher-order terms outside the jet flow. It will be assumed that the disturbance behavior is inviscid and that the mean flow is given, including all powers of ϵ, by

$$U=0\text{ and }V=\bar{V}(x_1) \tag{36}$$

The technique to be used is similar to that employed by Tam and Morris[17] for the stability of a compressible shear layer. The higher-order terms of the expansion Eq. (8) are given by solution of inhomogeneous equations of the form

$$\mathcal{L}_i(\phi_n)=\left(\frac{\partial}{\partial y^2}-k_0^2\right)\phi_0-\frac{k_0}{(k_0U_0-\omega)}\frac{\partial^2U_0}{\partial y^2}\phi_0$$
$$=\chi_{i,n}(x_1,y)\qquad n=1,2,\ldots \tag{37}$$

where the subscript i on the operators $\mathcal{L}$ and χ indicates that they are derived from the inviscid form of the disturbance equations. For $y\geq y_m$ $\chi_{i,n}$ takes a simple form, denoted by $\bar{\chi}_{i,n}$, and given by

$$\bar{\chi}_{i,n}(\phi_{n-1},\phi_{n-2},\phi_{n-3})=-2ik_0\frac{\partial}{\partial x_1}(\phi_{n-1})$$
$$+\left(\frac{\bar{V}}{\omega}\frac{\mathrm{d}k_0}{\mathrm{d}x_1}\frac{\partial}{\partial y}+\frac{2\bar{V}k_0}{\omega}\frac{\partial^2}{\partial x_1\partial y}-\frac{\partial^2}{\partial x_1^2}\right)(\phi_{n-2})$$
$$+\frac{i}{\omega}\left(\frac{\mathrm{d}^2\bar{V}}{\mathrm{d}x_1^2}-\bar{V}\frac{\partial^2}{\partial x_1^2}\right)\frac{\partial}{\partial y}(\phi_{n-3}) \tag{38}$$

Since the homogeneous form of Eq. (37) has an eigensolution, $\zeta(x_1,y)$, each solution for ϕ_n will consist of the sum of a particular integral and a complementary solution of the form $A_n(x_1)\zeta(x_1,y)$, where $A_n(x_1)$ may be determined from successive applications of the solvability condition for the existence of ϕ_{n+1}, that is,

$$\int_{-\infty}^{\infty} \zeta^*(x_1,y)\chi_{i,n+1}(x_1,y)\,dy=0, \qquad n=0,1,... \tag{39}$$

As shown here, this leads to an ordinary differential equation for $A_n(x_1)$.

The other linearly independent solution to the homogeneous form of Eq. (37) will be written as $\eta(x_1,y)$, which for $y \geq y_m$ takes the form

$$\eta(x_1,y)=\exp(k_0 y), \qquad y \geq y_m \tag{40}$$

The complete solution of Eq. (37) may then be obtained by applying the method of variation of parameters, giving

$$\phi_n(x_1,y)=A_n(x_1)\zeta(x_1,y) - \frac{\zeta(x_1,y)}{2k_0}\int_{-\infty}^{y}\eta(x_1,t)\chi_{i,n}(x_1,t)\,dt - \frac{\eta(x_1,y)}{2k_0}\int_{y}^{\infty}\zeta(x_1,t)\chi_{i,n}(x_1,t)\,dt \tag{41}$$

where the last term in Eq. (41) has been modified using the solvability condition Eq. (39), and noting that the homogeneous form of Eq. (37) is self-adjoint. Since the forms of $\zeta(x_1,y)$ and $\eta(x_1,y)$ are known for $y \geq y_m$ and $\chi_{i,n}=\bar{\chi}_{i,n}$, in this range the value of $\phi_n(x_1,y)$, after some manipulation, may be written as

$$\phi_n(x_1,y)=B_n(x_1)\exp(-k_0 y) - \frac{\exp(-k_0 y)}{2k_0}\int^{y}\exp(k_0 t)\bar{\chi}_{i,n}(x_1,t)\,dt - \frac{\exp(k_0 y)}{2k_0}\int_{y}^{\infty}\exp(-k_0 t)\bar{\chi}_{i,n}(x_1,t)\,dt \qquad y \geq y_m \tag{42a}$$

where

$$B_n(x_1)=A_n(x_1)-\frac{1}{2k_0}\left[\int_{y_m}\exp(k_0 t)\bar{\chi}_{i,n}(x_1,t)\,dt + \int_{-\infty}^{y_m}\exp(k_0 t)\chi_{i,n}(x_1,t)\,dt\right] \tag{42b}$$

Making use of Eq. (42a) it is readily shown that,

$$\phi_1(x_1,y)=(C_{10}+C_{11}y+C_{12}y^2)\exp(-k_0 y) \tag{43}$$

where

$$C_{10}(x_1)=B_1(x_1)+\frac{1}{2k_0}\frac{dA_0}{dx_1} \tag{44a}$$

$$C_{11}(x_1)=i\frac{dA_0}{dx_1} \tag{44b}$$

and

$$C_{12}(x_1)=-\frac{i}{2}A_0\frac{dk_0}{dx_1} \tag{44c}$$

After some lengthy algebra it may also be shown that

$$\phi_2(x_1,y)=(C_{20}+C_{21}y+C_{22}y^2+C_{23}y^3+C_{24}y^4)\exp(-k_0 y) \tag{45}$$

where all the coefficients C_{ij} are functions of x_1 and

$$C_{24}=\frac{i}{4}\frac{dk_0}{dx_1} \tag{46}$$

Continuing in the same manner it can be seen that for large values of y the multiple scales asymptotic expansion Eq. (8) may be written

$$\psi(x,y,t) \simeq \left\{\sum_{n=0}^{\infty} K_n \epsilon^n y^{2n}\right\}\exp[-k_0 y+i\theta(x,t)] \qquad y\rightarrow\infty \tag{47}$$

where the K_n are functions of x_1 only. It is clear that the asymptotic expansion does not hold for $y > \epsilon^{-1/2}$.

III. Results

Since the most conveniently and thus most often measured fluctuation in flow experiments is the axial velocity fluctuation, the calculations in this work are for this variable. This at least provides for possible comparison with experiment even if the question of what is meant by stability in a nonparallel flow will not be addressed; see Saric and Nayfeh[13] and Eagles and Weissman[9] for further discussion.

Numerical calculations are performed at several fixed frequencies. The frequencies are those whose neutral eigenvalues, according to a parallel flow approximation, were calculated by Ling and Reynolds.[4] These results provide convenient starting values for the present calculations. Since there is no characteristic length scale for the primary flow, the choice of Reynolds number to be used in the calculations is arbitrary. A value of $R=12.58$, corresponding to one of the local Reynolds numbers used by Ling and Reynolds,[4] is used in the present calculations. Since the nondimensionalization procedure, given in Eqs. (1a) and (1b), also depends on an arbitrary distance x_0^*, a specific example of the relationship between the subsequent calculations in this section and a hypothetical experiment will be given.

Consider a two-dimensional water jet issuing into stationary water. The kinematic viscosity of the water is taken to be $\nu^*=1.01\times10^{-6}$ m^2/s. The measured axial momentum flux of the jet is found to be, $J^*=1.69\times10^{-3}$ kg/s^2. For a density of $\rho^*=999$ kg/m^3, Eq. (2a) gives $\alpha^{*3}=9.48\times10^{-4}$ m^2/s$^{3/2}$. From Eqs. (1a) and (1b) the Reynolds number is related to x_0^* by

$$R=2\alpha^*\nu^{*-1/2}x_0^{*1/3} \tag{48}$$

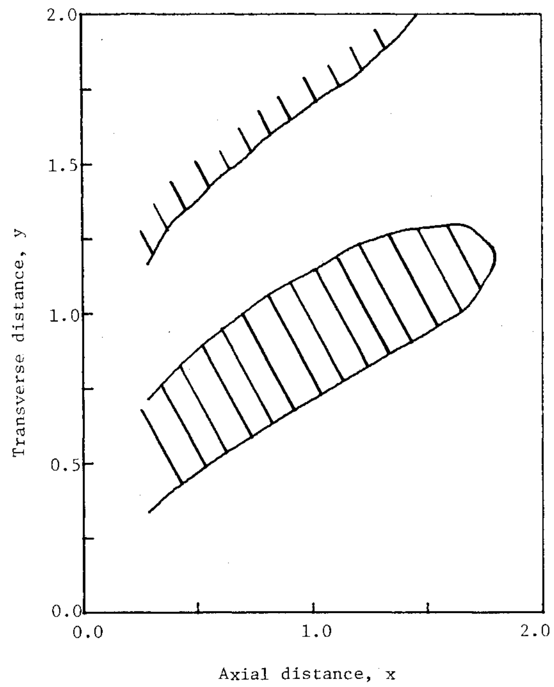

Fig. 1 Neutral curves for axial velocity fluctuations, $\sigma_u=0$ [Eq. (49)], $\omega=0.733$.

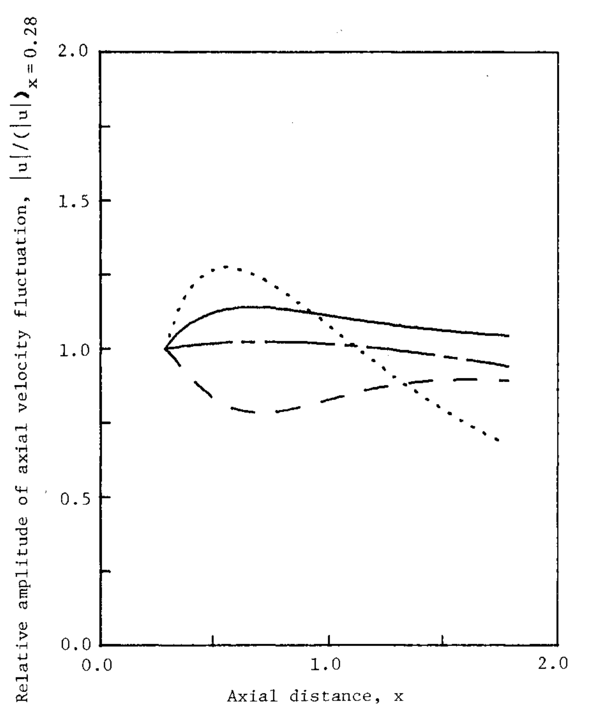

Fig. 2 Axial variation in amplitude of axial velocity fluctuations: ·····, $y=0.5$; - - -, $y=1.0$; ——, $y=1.5$; —— -, parallel flow calculations.

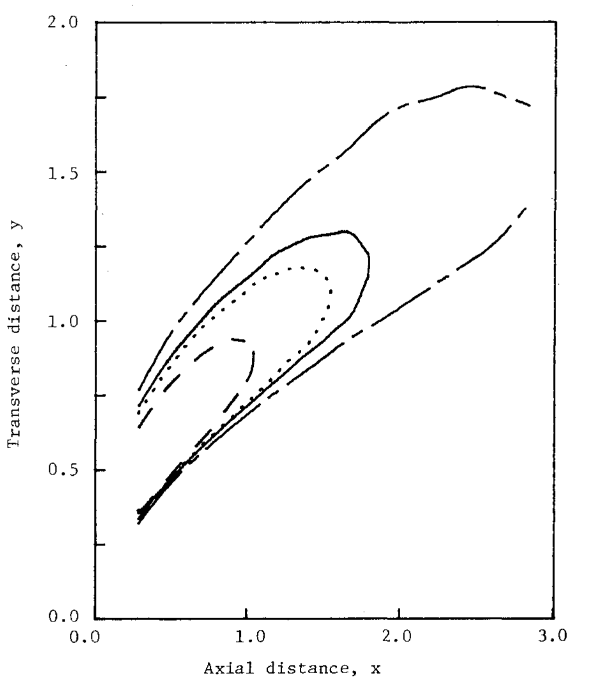

Fig. 3 Neutral curves for axial velocity fluctuations, $\sigma_u=0$: - - -, $\omega=1.059$; ·····, $\omega=0.813$; ——, $\omega=0.733$; —— -, $\omega=0.535$.

so that for $R=12.58$, $x_0^*=2.67\times 10^{-4}$ m. Then from Eqs. (1a) and (1b),

$$u_r^*=0.1 \text{ m/s}$$

and

$$\delta_r^*=1.27\times 10^{-4} \text{ m}$$

With these scales known the subsequent calculations may be related to their dimensional values.

The growth rate for the axial velocity fluctuation is given by

$$\sigma_u=\mathrm{Re}\left\{ik_0+\epsilon\left[ik_l+\frac{\partial}{\partial x_l}\ln\left(\frac{\partial\zeta}{\partial y}\right)\right]\right\} \tag{49}$$

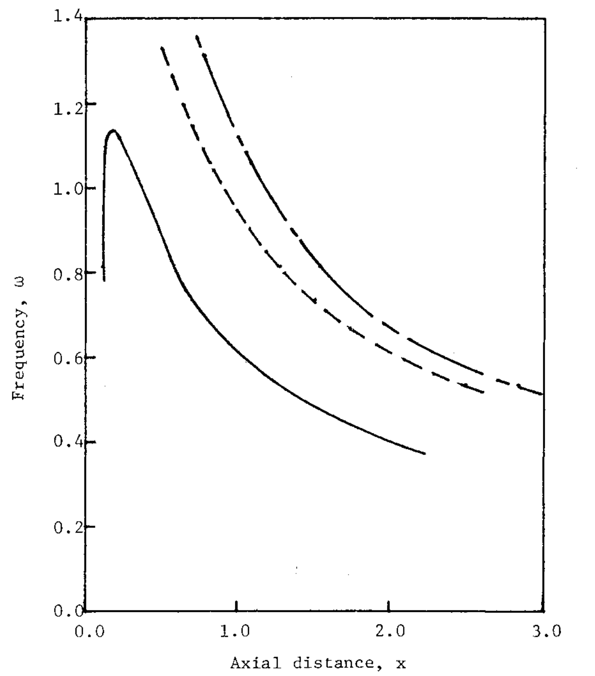

Fig. 4 Neutral curves of axial velocity fluctuation: ——, parallel flow approximation; - - -, $\mathrm{Re}\{ik_0+i\epsilon k_l\}=0$; —— -, $\sigma_u=0$ for all values of y.

Neutral stability curves for the axial velocity component, for which $\sigma_u=0$, are shown in Fig. 1 for a nondimensional frequency $\omega=\omega^*\delta_r^*/u_r^*=0.733$. For the hypothetical experiment described this corresponds to a dimensional frequency of 92 Hz. These curves have the characteristic features of curves for all computed frequencies. The shaded areas indicate growth of the axial velocity component with axial distance, the unshaded areas indicate decay. In light of the analysis given in the previous section the region of growth for large values of y may be regarded as not necessarily having physical significance. It is clear from Fig. 1 that at any transverse position, the growth rate and the wavenumber of the axial velocity will be different. This is shown in Fig. 2 where the relative axial behavior of the axial velocity fluctuation amplitude is shown at several transverse locations. It is clear that the variation of the amplitude of the axial velocity fluctuation is strongly dependent on the transverse position in the jet. The behavior, based on a locally parallel flow approximation is also shown for comparison. The parallel flow calculation indicates that the axial velocity fluctuation should decay at this frequency and for all transverse locations for $x>0.73$. The neutral contours for several other frequencies are shown in Fig. 3. As might be expected, the lower the frequency the greater the axial range over which the disturbance may grow at any transverse position. For the experiment described above a nondimensional frequency of $\omega=0.535$ corresponds to a real frequency of 67 Hz and the distance $x=3$ corresponds to 3.81×10^{-4} m.

As a final comparison between parallel and nonparallel flow calculations the neutral stability curve from parallel flow calculations is compared with those of nonparallel flow. The locally parallel flow calculations are performed for a fixed real frequency and constant Reynolds number. However, it should be noted that if a local velocity and length scale were used, then the local frequency would vary with x. Although in the case of parallel flow calculations there is no ambiguity about the definition and location of the neutral stability curve this is not so in the nonparallel case. Two nonparallel neutral curves are shown in Fig. 4. The first is a neutral curve based on $\mathrm{Re}\{ik_0+\epsilon ik_l\}$ being zero. As formerly argued this parameter has no physical significance since it ignores axial variations in the shape of the eigenfunctions. However, it was chosen by Garg and Round[11] to describe nonparallel effects on jet stability. It should be noted that direct comparison of the present measurements with those of Garg and Round is difficult since they presented their calculations for a fixed, but arbitrary, axial location for various Reynolds numbers. The second neutral curve is determined as the axial limit for growth of the axial velocity fluctuation at any transverse position. Points on this curve may be obtained from Fig. 3. The nonparallel neutral curves clearly indicate disturbance growth over a wider range of frequency and axial distance than that predicted by parallel flow calculations. In order to compare these stability calculations with experimental data it would be necessary to measure axial velocity disturbance growth rates and wavenumbers at various transverse locations. Previous investigations of the stability of a two-dimensional jet, such as Sato and Sakao,[18] do not provide sufficient detail to enable a valid comparison with the present calculations.

IV. Summary and Conclusions

The method of multiple scales has been used to examine the linear stability of a two-dimensional jet. The calculations indicate that the axial velocity fluctuation grows over a greater region of the jet flow than that predicted by parallel-flow theory. The behavior of the disturbance is seen to be a strong function of transverse location. It has also been shown that the method of multiple scales, as presently applied to linear flow stability problems, gives an invalid solution at large distances from the sheared region of the flow. This is seen to lead to physically unrealistic results. However, it should be noted that since the parameter ϵ is inversely proportional to the local Reynolds number of the flow this problem does not invalidate the calculation procedure, for practical purposes, for high Reynolds number flows such as boundary layers. Caution should be used in applying the technique to examine the stability of free shear flows which are inherently unstable at much lower Reynolds numbers. Finally, the calculations clearly show that considerable care should be exercised in comparing experimental stability measurements with numerical calculations if a valid comparison is to be made.

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